NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Mathematics Assessment Task 3
Term 2, 2008

Name:_______________________________ **Mathematics Class:_________**

Time Allowed: 60 minutes + 2 minutes reading time

Available Marks: 55

Instructions:

- Questions are of equal value.
- Start each question on a new page.
- Put your name on the top of each page.
- Attempt all five questions.
- Show all necessary working.
- Marks may be deducted for incomplete or poorly arranged work.
- Write on one side of the page only. Do not work in columns.
- Each question will be collected separately.
- If you do not attempt a question, submit a blank page with your name and the question number clearly displayed.

Question	1abcd	1e	2a	2bc	3	4ab	4c	5a	5bc	Total	
H3	/9									/9	
H5		/2	/4		/11		/4			/21	
H8				/7		/7			/6	/20	
H9								/5		/5	
										/55	%

Question 1 (11 marks)

a) If $x = 0 \cdot 2$, evaluate e^{x^2} to 3 significant figures. **2**

b) Neatly sketch $y = e^{-x}$ showing all key features. **1**

c) Differentiate with respect to x:

i) $y = 2e^{-x}$ **1**

ii) $y = \dfrac{e^{2x}}{x+3}$ **2**

d) Find the equation of the tangent to the curve $y = e^x + 1$ at the point $(1,\ e+1)$ **3**

e) Evaluate $\displaystyle\int_1^2 (x^2 + 7)\ dx$ **2**

Question 2 (11 marks) Start a new page.

a) Find the indefinite integrals of

i) $x^2(x-3)$ **2**

ii) $(2x+7)^{10}$ **2**

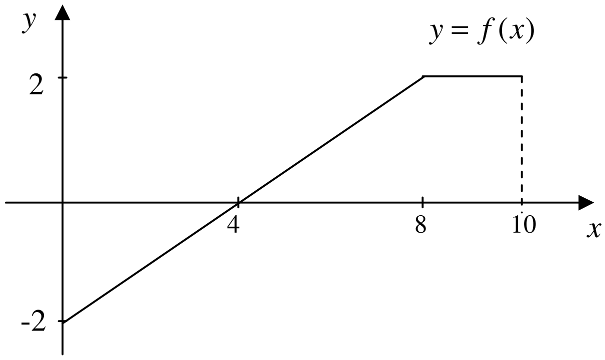
b) i) Sketch the curve $y = 2x^3$ in the domain $-2 \le x \le 2$ **1**

ii) Find the area bounded by the curve $y = 2x^3$, the x axis, $x = 2$ and $x = -2$. **3**

Question 2 continued over page…

Question 2 continued.

c)



Using the graph above, evaluate:

i) $\int_{4}^{10} f(x)\, dx$ **1**

ii) $\int_{0}^{4} f(x)\, dx$ **1**

iii) $\int_{0}^{10} f(x)\, dx$ **1**

Question 3 (11 marks) **Start a new page.**

a) The first term of a geometric series is 8 and the infinite sum is 64. Find the common ratio. **2**

b) Evaluate $\sum_{n=1}^{5}(-5)^{n-1}$ **2**

c) Find the sum of the first 10 terms of the series $2+2\sqrt{5}+10+...$ **2**

d) Mr Jordan invested \$500 at the start of this year into a bank account paying $7\cdot5\%$ interest per annum, compounding yearly. He plans to invest \$500 into this account at the start of every year.

i) Write an expression involving the sum of a series which will give the total value of the investment immediately after the 10^{th} deposit of \$500. **2**

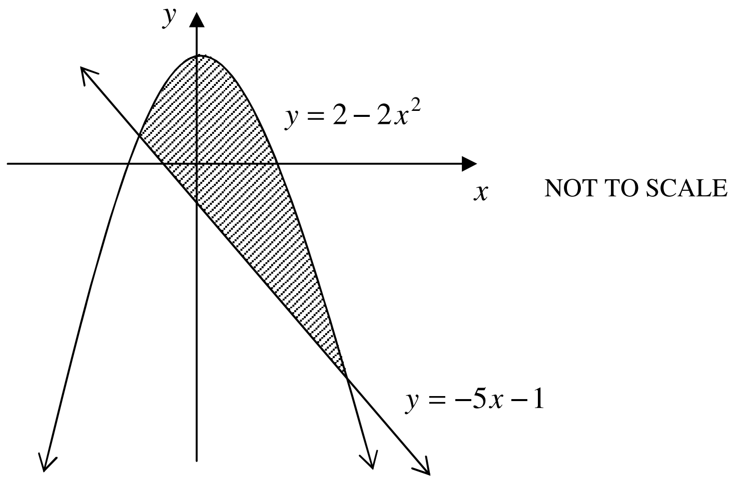
ii) Thus calculate the balance of the bank account at this time, correct to the nearest dollar. **1**

e) By considering it as a geometric series, express $0\cdot\dot{3}\dot{2}$ as a fraction in its simplest form. **2**

Question 4 (11 marks) **Start a new page.**

a) Find the exact volume of the solid formed when the area bounded by the curve $y^2 = 8x$, the y axis and the line $y = 4$ is rotated about the y axis. **3**

b) The diagram below shows the graphs of the parabola $y = 2 - 2x^2$ and the straight line $y = -5x - 1$.



i) Find the x coordinates of the points of intersection of the two curves. **1**

ii) Find the area of the region enclosed by the two curves. **3**

c) i) Show that $\dfrac{x}{(x+3)^3} = \dfrac{1}{(x+3)^2} - \dfrac{3}{(x+3)^3}$ **1**

ii) Hence, or otherwise, evaluate $\displaystyle\int_0^3 \frac{x}{(x+3)^3}\, dx$ **3**

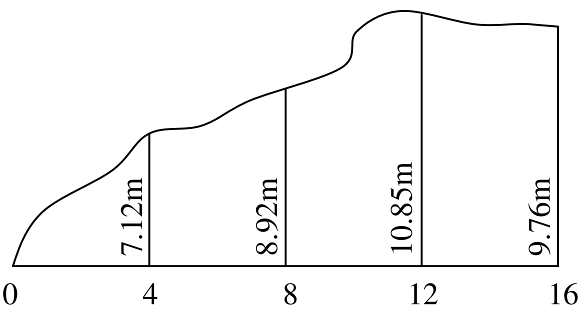
Question 5 (11 marks) **Start a new page.**

a) Miss James is paying off her Christmas trip to Europe. She borrowed \$12 000 and is being charged 15% p.a. interest, compounding monthly. She makes equal quarterly repayments, the first being made after 3 months. Let Q be the quarterly repayment.

i) What is the balance owing, to the nearest dollar, immediately before she makes her first repayment? **2**

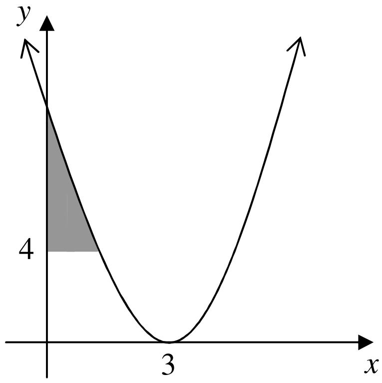
ii) If she has agreed to pay the loan off over 5 years, what is the amount of each repayment she makes? **3**

b) Using all of the measurements below, a surveyor calculated the approximate area of this irregular block of land.



Calculate the approximate area using Simpson's rule. **3**

c) The diagram below shows the parabola $y = (x-3)^2$. Find the area of the shaded region. **3**



End of paper.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

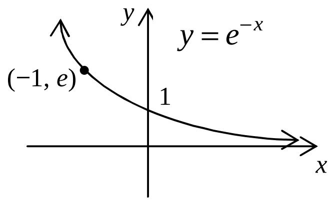
Solutions - Year 12 Mathematics Assessment Task 3, Term 2 2008

Question 1

a) $e^{0\cdot 04} = 1\cdot 0408107...$

$= 1\cdot 04 \; (3sf)$

b)



c) i $y = 2e^{-x}$

$\frac{dy}{dx} = -2e^{-x}$

ii $y = \frac{e^{2x}}{x+3}$ $\quad u = e^{2x} \quad v = x+3$

$u' = 2e^{2x} \quad v' = 1$

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$$

$$= \frac{2e^{2x}(x+3) - e^{2x}}{(x+3)^2}$$

$$= \frac{2xe^{2x} + 5e^{2x}}{(x+3)^2}$$

d) $y = e^x + 1 \qquad (1,\, e+1)$

$\frac{dy}{dx} = e^x$

when $x = 1, \frac{dy}{dx} = e, \therefore$ gradient of tangent = e

$y - y_1 = m(x - x_1)$

$\therefore y - (e+1) = e(x-1)$

$y = ex + 1$

e) $$\int_1^2 \left(x^2 + 7\right) dx = \left[\frac{x^3}{3} + 7x\right]_1^2$$

$$= \frac{2^3}{3} + 7(2) - \left(\frac{1}{3} + 7\right)$$

$$= \frac{28}{3} = 9\tfrac{1}{3}$$

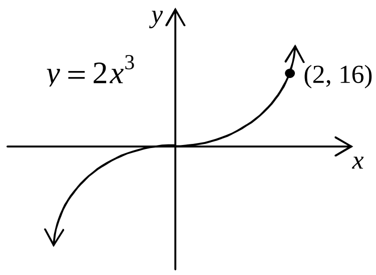
Question 2

a) i $$\int x^2(x-3)\,dx = \int (x^3 - 3x^2)\,dx$$

$$= \frac{x^4}{4} - x^3 + C$$

ii $$\int (2x+7)^{10}\,dx = \frac{(2x+7)^{11}}{22} + C$$

b) i



ii Function is odd, so area is symmetrical about y axis

$$A = 2 \times \int_0^2 2x^3 dx$$

$$= 2\left[\frac{x^4}{4}\right]_0^2$$

$$= 2 \times \frac{32}{2}$$

$$= 16 \text{ units}^2$$

c) i From graph, $A = 2\times 2 + \frac{1}{2}\times 4 \times 2$

$= 8$

$$\therefore \int_4^{10} f(x)\,dx = 8$$

ii $A = \frac{1}{2}\times 4\times 2 = 4$

But area is below x axis

$$\therefore \int_0^4 f(x)\,dx = -4$$

iii $$\int_0^{10} f(x)\,dx = \int_4^{10} f(x)\,dx + \int_0^4 f(x)\,dx$$

$$= 8 - 4$$

$$= 4$$

Question 3

a)

$$S_\infty = \frac{a}{1-r} \qquad a = 8,\ \ S_\infty = 64$$

$$64 = \frac{8}{1-r}$$

$$1 - r = \frac{1}{8}$$

$$\therefore r = \frac{7}{8}$$

b)

$$\sum_{n=1}^{5}(-5)^{n-1} \longrightarrow \text{G.S. with } a = 1,\ r = -5,\ n = 5$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_5 = \frac{1\left(1-(-5)^5\right)}{1+5}$$

$$= \frac{1+3125}{6}$$

$$= 521$$

c)

$$2 + 2\sqrt{5} + 10 + \ldots. \longrightarrow \text{G.S with } a = 2,\ r = \sqrt{5},\ n = 10$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$\therefore S_{10} = \frac{2\left(\left(\sqrt{5}\right)^{10} - 1\right)}{\sqrt{5} - 1}$$

$$= \frac{2\left(5^5 - 1\right)}{\sqrt{5} - 1}$$

$$= \frac{6428\left(\sqrt{5}+1\right)}{4}$$

$$= 1607\left(\sqrt{5}+1\right)$$

d) i Let A_n be the value of the investment at the start of year n.

$$A_1 = 500$$

$$A_2 = 500 \times 1\cdot075 + 500$$

$$A_3 = 500 \times 1\cdot075^2 + 500 \times 1\cdot075 + 500$$

A_{10} gives the value of the investment after the 10th deposit

$$A_{10} = 500 \times 1\cdot075^{10} + 500 \times 1\cdot075^{9} + \ldots + 500$$

This is a G.S. with $a = 500$, $r = 1\cdot075$, $n = 10$

$$\therefore A_{10} = \frac{500(1\cdot075^{10} - 1)}{0\cdot075}$$

ii) $A_{10} = 7073.543\ldots \approx \$7074$

e)

$$0\cdot\dot{3}\dot{2} = 0\cdot32 + 0\cdot0032 + 0\cdot000032 + \ldots$$

Gives a G.S. with $a = 0\cdot32$, $r = 0\cdot01$

Since $|r| < 1$ a limiting sum exists

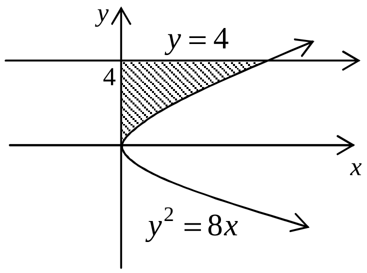
$$S_\infty = \frac{a}{1-r} = \frac{0\cdot32}{1-0\cdot01}$$

$$= \frac{0\cdot32}{0\cdot99}$$

$$= \frac{32}{99}$$

Question 4

a)



Volume around y axis is given by $V = \pi\int_a^b x^2\,dy$

$$x = \frac{y^2}{8}$$

$$\therefore x^2 = \frac{y^4}{64}$$

$$\therefore V = \pi\int_0^4 \frac{y^4}{64}\,dy$$

$$= \pi\left[\frac{y^5}{320}\right]_0^4$$

$$= \frac{1024\pi}{320}$$

$$= \frac{16\pi}{5} \text{ units}^3$$

b) i Solving simultaneously,

$$2 - 2x^2 = -5x - 1$$

$$0 = 2x^2 - 5x - 3$$

$$0 = (x-3)(2x+1)$$

$$\therefore x = 3, -\tfrac{1}{2}$$

ii

$$\text{Area} = \int_{-\frac{1}{2}}^{3}\left(2 - 2x^2\right)dx - \int_{-\frac{1}{2}}^{3}\left(-1 - 5x\right)dx$$

$$= \int_{-\frac{1}{2}}^{3}\left(3 - 2x^2 + 5x\right)dx$$

$$= \left[3x - \frac{2x^3}{3} + \frac{5x^2}{2}\right]_{-\frac{1}{2}}^{3}$$

$$= \left[3(3) - \frac{2(3)^3}{3} + \frac{5(3)^2}{2}\right] - \left[3\left(-\tfrac{1}{2}\right) - \frac{2\left(-\frac{1}{2}\right)^3}{3} + \frac{5\left(-\frac{1}{2}\right)^2}{2}\right]$$

$$= 9 - 18 + \frac{45}{2} + \frac{3}{2} - \frac{1}{12} - \frac{5}{8}$$

$$= \frac{343}{24} \text{ units}^2$$

c) i

$$\text{RHS} = \frac{1}{(x+3)^2} - \frac{3}{(x+3)^3}$$

$$= \frac{x+3-3}{(x+3)^3}$$

$$= \frac{x}{(x+3)^3} = \text{LHS}$$

ii

$$\int_0^3 \frac{x}{(x+3)^3}\,dx = \int_0^3\left(\frac{1}{(x+3)^2} - \frac{3}{(x+3)^3}\right)dx$$

$$= \left[\frac{-1}{x+3} + \frac{3}{2(x+3)^2}\right]_0^3$$

$$= -\frac{1}{6} + \frac{3}{72} + \frac{1}{3} - \frac{3}{18}$$

$$= \frac{1}{24}$$

Question 5

a) i

15% p.a. = $1\cdot 25\%$ p.m. = $0\cdot 0125$ p.m. Principal = \$12000

Repayments, Q, are made at the end of every 3 months.

Let A_n be the amount owing after n months.

$$A_1 = 12000\times 1\cdot 0125$$
$$A_2 = A_1\times 1\cdot 0125$$
$$= 12000\times 1\cdot 0125^2$$
$$A_3 = A_2\times 1\cdot 0125 - Q$$
$$= 12000\times 1\cdot 0125^3 - Q$$

$\therefore$ amount owed before first payment = $12000\times 1\cdot 0125^3$

$$= 12455\cdot 648...$$
$$= \$12456 \text{ (nearest dollar)}$$

ii

Continuing the series,

$$A_4 = A_3\times 1\cdot 0125$$
$$= \left(12000\times 1\cdot 0125^3 - Q\right)1\cdot 0125$$
$$A_5 = A_4\times 1\cdot 0125$$
$$= \left(12000\times 1\cdot 0125^3 - Q\right)1\cdot 0125^2$$
$$A_6 = A_5\times 1\cdot 0125 - Q$$
$$= \left(12000\times 1\cdot 0125^3 - Q\right)1\cdot 0125^3 - Q$$
$$= 12000\times 1\cdot 0125^6 - 1\cdot 0125^3 Q - Q$$

After 5 years loan is paid. 20 repayments have been made.

$$A_{60} = 12000\times 1\cdot 0125^{60} - 1\cdot 0125^{57}Q - 1\cdot 0125^{54}Q - ... - Q$$

terms 2 onwards form a G.S. with $a = Q,\ r = 1\cdot 0125^3,\ n = 20$

$A_{60} = 0$ since load is repaid,

$$\therefore 0 = 12000\times 1\cdot 0125^{60} - \frac{R\left(1\cdot 0125^{60} - 1\right)}{1\cdot 0125^3 - 1}$$
$$R = \frac{12000\times 1\cdot 0125^{60}\left(1\cdot 0125^3 - 1\right)}{\left(1\cdot 0125^{60} - 1\right)}$$
$$= 867.187....$$

$\therefore$ quarterly repayments are \$867 (nearest dollar)

b) 5 function values = 2 applications of Simpson's Rule

$$\int_0^{16} f(x)dx = \int_8^{16} f(x)dx + \int_0^8 f(x)dx$$
$$= \tfrac{8}{6}\left(0 + 4(7\cdot 12) + 8\cdot 92\right) + \tfrac{8}{6}\left(8\cdot 92 + 4(10\cdot 85) + 9\cdot 76\right)$$
$$= 132\cdot 64\text{m}^2$$

c) $y = (x-3)^2$ $\quad y$ intercept $= 9$, $\therefore$ Area = $\int_4^9 x\,dy$

need to make x the subject: $\pm\sqrt{y} = x - 3$

$$x = 3 \pm \sqrt{y}$$

from graph, equation required is $x = 3 - \sqrt{y}$

$$\therefore \text{A} = \int_4^9 \left(3 - \sqrt{y}\right) dy$$
$$= \left[3y - \frac{2y^{\frac{3}{2}}}{3}\right]_4^9$$
$$= 27 - 18 - \left(12 - \frac{16}{3}\right)$$
$$= \frac{7}{3}\text{units}^2$$